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Uniform approximation theorems for real-valued continuous functions

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Abstract

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For a topological space X, $F(X)$ denotes the algebra of real-valued functions over X and $C(X)$ the subalgebra of all functions in $F(X)$ which are continuous. In this paper we characterize the uniformly dense linear subspaces of $C(X)$ by means of the so-called "Lebesgue chain condition". This condition is a generalization to the unbounded case of the S-separation by Blasco and Moltó for the bounded case. Through the Lebesgue chain condition we also characterize the linear subspaces of $F(X)$ whose uniform closure is closed under composition with uniformly continuous functions.

Keywords: Uniform approximation, Lebesgue chain, Lebesgue chain condition, property C, cozero-sets, star-finite cover, 2-finite cover, uniformly continuous.

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1. Introduction

For a completely regular space X, $C(X)$ and $C^*(X)$ denote, respectively, the algebra of all real-valued continuous functions and bounded real-valued continuous functions over X. When X is not a pseudocompact space, i.e., if $C^*(X) \neq C(X)$, theorems about uniform density for subsets of $C^*(X)$ are not directly translatable to $C(X)$. In [1], Anderson gives a sufficient condition in order that certain rings of $C(X)$ be uniformly dense, but this condition is not necessary.

In this paper we study the uniform closure of a linear subspace of real-valued functions and we obtain, in particular, a necessary and sufficient condition of uniform

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density in $C(X)$. These results generalize, for the unbounded case, those obtained by Blasco and Moltó for the bounded case [2]. The approximation technique used by them (essentially the same as Tietze [9], Mrowka [7] and Jameson [6]) is also the starting point for us.

In order to establish their results, Blasco and Moltó define a new concept, the S-separation, which is a suitable debilitation of that of complete separation. Here, we introduce a parallel concept, namely the "Lebesgue chain condition". From it we obtain our density theorem for linear subspaces of $C(X)$. Following the same structure of [2], we define the "property C". This property agrees in the bounded case with the property S of Blasco and Moltó. But, although the property S permits to characterize the linear subspaces (of bounded functions) whose uniform closure is a ring or a lattice containing all the real constant functions, the property C does not permit such characterization in the unbounded case. Nevertheless, by means of the property C we shall be able to characterize the linear subspaces of $F(X)=\mathbb{R}^X$ whose uniform closure is closed under composition with uniformly continuous functions over $\mathbb{R}$. This enables us to put some examples and to obtain some results such as the following, "every uniformly closed ring and lattice of real-valued functions containing all the real constant functions is closed under composition with a large class of continuous functions over $\mathbb{R}$". We shall show that such class contains $C^*(\mathbb{R})$ but, in general, it is not $C(\mathbb{R})$ (Proposition 4.9 and Example 4.10).

2. Uniform approximation for linear subspaces

For a set X, $F(X)$ (respectively $F^*(X)$) denotes the class of all (respectively all bounded) real-valued functions over X. $F(X)$ is a linear space by introducing pointwise addition and scalar multiplication. We let $\mathbb{R}$ be the set of real numbers, $\mathbb{Q}$ the rational numbers, $\mathbb{Z}$ the integer numbers, $\mathbb{N}$ the positive integer numbers and $\bar{\mathbb{R}}=\mathbb{R}\cup\{-\infty,\infty\}$. Given $f\in F(X)$ and a real number α, we let $L_\alpha(f)=\{x\in X: f(x)\leqslant\alpha\}$ and $L^\alpha(f)=\{x\in X: f(x)\geqslant\alpha\}$. We refer to $L_\alpha(f)$ and $L^\alpha(f)$ as the Lebesgue sets of f.

2.1. Definition. Let f be a function from $F(X)$. A *Lebesgue chain* of f we define as any countable cover $\{C_n\}_{n\in\mathbb{Z}}$ of X such that

$$C_n=\{x\in X:\ \alpha_{n-1}<f(x)<\alpha_{n+1}\}$$

and $\{\alpha_n\}_{n\in\mathbb{Z}}$ is a nondecreasing sequence in $\bar{\mathbb{R}}$ for which there exists $r>0$ satisfying $\alpha_n-\alpha_{n-1}\geqslant r$ provided α_n is a real number.

2.2. Definition. A set $\mathscr{F}$ of $F(X)$ satisfies the *Lebesgue chain condition* for a function f if there exists $k>0$ such that for every Lebesgue chain $\{C_n\}_{n\in\mathbb{Z}}$ of f there is $g\in\mathscr{F}$ such that $|g(x)-n|<k$ when $x\in C_n$.

2.3. Proposition. *Let $\mathscr{F} \subset F(X)$ a linear subspace over $\mathbb{R}$ and let $f \in F(X)$. If $\mathscr{F}$ satisfies the Lebesgue chain condition for f, then f belongs to the uniform closure of $\mathscr{F}$.*

Proof. Let $\varepsilon > 0$, $\alpha_n = n\varepsilon$ and let $\{C_n\}_{n\in\mathbb{Z}}$ be the Lebesgue chain of f given by the sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$. By hypothesis, there exists $g \in \mathscr{F}$ with $|g(x)-n| < k$ if $x \in C_n$. Consequently $|\varepsilon g(x) - \varepsilon n| < \varepsilon k$ if $x \in C_n$ and

$$|\varepsilon g(x) - f(x)| \leq |\varepsilon g(x) - \varepsilon n| + |f(x) - \varepsilon n| < (k+1)\varepsilon.$$

Since $\{C_n\}_{n\in\mathbb{Z}}$ is a cover of X, then $|\varepsilon g(x) - f(x)| < (k+1)\varepsilon$ for every $x \in X$ and hence $f \in \bar{\mathscr{F}}$ (uniform closure of $\mathscr{F}$). □

A set $\mathscr{F} \subset F(X)$ S-separates the Lebesgue sets of a function f if for every $\delta > 0$ and $\alpha < \beta$ there exists $g \in \mathscr{F}$ such that $0 \leq g \leq 1$, $g(L_\alpha(f)) \subset [0, \delta]$ and $g(L^\beta(f)) \subset [1-\delta, 1]$. If the above function g satisfies the less restrictive conditions $-\delta \leq g \leq 1+\delta$, $g(L_\alpha(f)) \subset [-\delta, \delta]$ and $g(L^\beta(f)) \subset [1-\delta, 1+\delta]$, we shall say that $\mathscr{F}$ S'-separates the Lebesgue sets of f.

The next proposition (proved in [4]) gives the equivalence, for the bounded case, among S-separation, S'-separation and Lebesgue chain condition.

2.4. Proposition (Garrido and Montalvo [4]). *Let $\mathscr{F} \subset F^*(X)$ a linear subspace over $\mathbb{R}$ and let $f \in F^*(X)$. The following conditions are equivalent:*

(a) *$\mathscr{F}$ satisfies the Lebesgue chain condition for f.*

(b) *$\mathscr{F}$ S'-separates the Lebesgue sets of f.*

(c) *$\mathscr{F}$ S-separates the Lebesgue sets of f.*

2.5. Corollary (Blasco and Moltó [2]). *Let $\mathscr{F} \subset F^*(X)$ a linear subspace over $\mathbb{R}$ and let $f \in F^*(X)$. If $\mathscr{F}$ S-separates the Lebesgue sets of f, then f can be uniformly approximated by members of $\mathscr{F}$.*

A zero-set (respectively a cozero-set) in a topological space X is a set of the form $Z(f) = \{x \in X: f(x) = 0\}$ (respectively $\mathrm{coz}(f) = X \backslash Z(f)$) with $f \in C(X)$. It is well known that countable intersection (respectively countable union) of zero-sets (cozero-sets) is a zero-set (cozero-set) (Gillman and Jerison [5]).

2.6. Definition. We say that a countable cover $\{C_n\}_{n\in\mathbb{Z}}$ of X is a *2-finite cover* if $C_n \cap C_m = \emptyset$ when $|n - m| > 1$.

2.7. Theorem. *Let X be a topological space and let $\mathscr{F} \subset C(X)$ a linear subspace over $\mathbb{R}$. The following conditions are equivalent:*

(a) *$\mathscr{F}$ is uniformly dense in $C(X)$.*

(b) *For each countable 2-finite cover $\{C_n\}_{n\in\mathbb{Z}}$ of X by cozero-sets there is $h \in \mathscr{F}$ with $|h(x) - n| < 2$ when $x \in C_n$ $(n \in \mathbb{Z})$.*

Proof. (a)$\Rightarrow$(b) Let $\{C_n\}_{n\in\mathbb{Z}}$ be a 2-finite cover of X by cozero-sets and let $f_n \in C(X)$ a nonnegative function with $\operatorname{coz}(f_n) = C_n$ for every $n \in \mathbb{Z}$. Put $g_n = f_n/(\sum_{n\in\mathbb{Z}} f_n)$. It is easy to check that $\{g_n\}_{n\in\mathbb{Z}}$ is a partition of unity on X and $\operatorname{coz}(g_n) = C_n$. Now, let $g = \sum_{n\in\mathbb{Z}} ng_n$. Since $\mathscr{F}$ is uniformly dense in $C(X)$, then there exists $h \in \mathscr{F}$ such that $|h - g| < 1$. Thus h is the required function, because if $x \in C_n$, then

$$|g(x) - n| = |g_{n+1}(x) - g_{n-1}(x)| \leq 1$$

and hence $|h(x) - n| \leq |h(x) - g(x)| + |g(x) - n| < 2$.

(b)$\Rightarrow$(a) It follows from Proposition 2.3 because, according to the hypothesis (b), $\mathscr{F}$ satisfies the Lebesgue chain condition for every $f \in C(X)$. $\square$

Remark. Obviously the number 2 in the above theorem can be replaced by any scalar $k > 1$.

2.8. Examples. (1) Let X be an open set of $\mathbb{R}^n$ and $\mathscr{F} = C^\infty(X) =$ {infinitely differentiable functions over X}. By applying Theorem 2.7 we shall prove that $\mathscr{F}$ is uniformly dense in $C(X)$.

It is well known that for every countable cover $\{C_n\}_{n\in\mathbb{Z}}$ of X by cozero-sets, there exists a locally finite partition of unity $\{h_n\}_{n\in\mathbb{Z}}$ on X, by functions of $C^\infty(X)$ subordinated to it [8]. Now, if $C_n \cap C_m = \emptyset$ when $|n - m| > 1$ and $h = \sum_{n\in\mathbb{Z}} nh_n$, then it is easy to check that $|h(x) - n| < 2$ if $x \in C_n$ and therefore $\mathscr{F}$ is uniformly dense in $C(X)$.

(2) Let $\mathscr{F}$ be the linear subspace of $C(\mathbb{N})$ generated by the functions taking integer values. It is easy to prove that $\mathscr{F}$ is not $C(\mathbb{N})$ and from Theorem 2.7 $\mathscr{F}$ is uniformly dense in $C(\mathbb{N})$.

3. On a theorem of Anderson

In [1], Anderson gives a sufficient but not necessary condition of uniform density for a divisible subring of $C(X)$. A subring $\mathscr{F}$ is divisible in case that for every $f \in \mathscr{F}$ and $n \in \mathbb{Z}$ there is $g \in \mathscr{F}$ such that $f = ng$. The proof of Anderson is very complicate and now we shall prove the same result as a consequence of Theorem 2.7, because this theorem remains valid when $\mathscr{F}$ is a linear subspace over $\mathbb{Q}$, just in case of a divisible subring. Previously we need to state the following lemma.

3.1. Lemma. *Let X be a topological space and let $\{C_n\}_{n=0}^\infty$ be a 2-finite cover of X by cozero-sets. If $\mathscr{F}$ is a ring that S_1-separates (completely separates) every pair of disjoint zero-sets in X (i.e., if Z_0 and Z_1 are disjoint zero-sets in X, there is $f \in \mathscr{F}$ with $0 \leq f \leq 1$, taking the values 0 and 1 on Z_0 and Z_1 respectively). Then there exists a partition of unity $\{g_n\}_{n=0}^\infty$ on X by functions of $\mathscr{F}$ with $\operatorname{coz}(g_n) \subset C_n$ for each n.*

Proof. Firstly, according to these hypotheses, note that the sets $D_n = (\bigcup_{k=0}^{n} C_k)\setminus C_{n+1}$, $n = 0, 1, \ldots$ are zero-sets in X because $D_n = \bigcap_{k=n+1}^{\infty} (X\setminus C_k)$. Let $g_0 \in \mathcal{F}$ with $0 \leq g_0 \leq 1$, $g_0(D_0) = 1$ and $g_0(X\setminus C_0) = 0$. Obviously $\operatorname{coz}(g_0) \subset C_0$.

Assume that $g_0, g_1, \ldots, g_{n-1}$ has been constructed in $\mathcal{F}$ such that $g_0(x) + \cdots + g_{n-1}(x) = 1$ when $x \in D_{n-1}$ and $\operatorname{coz}(g_i) \subset C_i$ $(i = 0, \ldots, n-1)$. Let $u \in \mathcal{F}$ with $0 \leq u \leq 1$, $u(D_n) = 1$ and $u(X\setminus\bigcup_{i=0}^{n} C_i) = 0$ and define $g_n = u - u\sum_{i=0}^{n-1} g_i$. It is clear that $g_n \in \mathcal{F}$, $\operatorname{coz}(g_n) \subset C_n$ because $g_n(x) \neq 0$ when $u(x) \neq 0$ and $\sum_{i=0}^{n-1} g_i(x) \neq 1$ and $\sum_{i=0}^{n} g_i(x) = 1$ when $x \in D_n$ because $\sum_{i=0}^{n} g_i(x) = u(x) + (1 - u(x))\sum_{i=0}^{n-1} g_i(x)$. In this way, the sequence $\{g_n\}_{n=0}^{\infty}$ is a partition of unity on X with $\operatorname{coz}(g_n) \subset C_n$. $\square$

A cover $\mathcal{D}$ of X is a star-finite cover if and only if each member of $\mathcal{D}$ meets at most finitely many members of $\mathcal{D}$.

3.2. Theorem (Anderson [1]). *Let X be a topological space and let $\mathcal{F}$ be a divisible subring of $C(X)$ which satisfies*:

(i) *$\mathcal{F}$ S_1-separates every pair of disjoint zero-sets in X.*

(ii) *For every sequence $\{f_n\}_{n=0}^{\infty}$ of nonnegative functions in $\mathcal{F}$ such that $\{\operatorname{coz}(f_n)\}_{n=0}^{\infty}$ is a star-finite cover of X, $\sum_{n=0}^{\infty} f_n \in \mathcal{F}$.*

Under these conditions $\mathcal{F}$ is uniformly dense in $C(X)$.

Proof. Let $\{C_n\}_{n\in\mathbb{Z}}$ be a 2-finite cover of X by cozero-sets. Applying the previous lemma to the 2-finite covers of X by cozero-sets:

(1) $C'_0 = \bigcup_{k\leq 0} C_k$ and $C'_n = C_n$ $(n > 0)$,

(2) $C''_0 = \bigcup_{k\geq 0} C_k$ and $C''_n = C_{-n}$ $(n > 0)$,

we obtain two partitions of unity $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$ on X by functions of $\mathcal{F}$, with $\operatorname{coz}(u_n) \subset C'_n$ and $\operatorname{coz}(v_n) \subset C''_n$ for every n. From (ii), the functions $g_1 = u_0 + \sum_{n\geq 0} nu_n$ and $g_2 = v_0 + \sum_{n\geq 0} nv_n$ belong to $\mathcal{F}$ and then $h = v_0 g_1 - u_0 g_2$ is also in $\mathcal{F}$.

This function h satisfies $|h(x) - n| \leq 1$ if $x \in C_n$ $(n \in \mathbb{Z})$. If $x \in C_n$ with $n > 0$, then $u_0(x) = 0$, $v_0(x) = 1$ and $h(x) = (n-1)u_{n-1}(x) + nu_n(x) + (n+1)u_{n+1}(x) = n + u_{n+1}(x) - u_{n-1}(x)$. Similarly when $x \in C_n$ with $n \leq 0$. $\square$

Remark. The conditions in the above theorem are not necessary. For instance, if $\mathcal{F}$ is the set of all the functions in $C(\mathbb{R})$ which have a derivate in all but finitely many points of $\mathbb{R}$, then $\mathcal{F}$ is a divisible subring uniformly dense in $C(\mathbb{R})$ but $\mathcal{F}$ does not satisfy the hypothesis (ii) in the theorem.

4. Characterization of the Lebesgue chain condition. The property C

In the following proposition we give a useful characterization of the Lebesgue chain condition.

4.1. Proposition. *Let $\mathscr{F} \subset F(X)$ a linear subspace over $\mathbb{R}$ and let $f \in \mathscr{F}$. The following conditions are equivalent*:

(a) *$\mathscr{F}$ satisfies the Lebesgue chain condition for f.*

(b) *$\varphi \circ f \in \bar{\mathscr{F}}$ for every uniformly continuous and monotonic function φ over $\mathbb{R}$.*

(c) *$\varphi \circ f \in \bar{\mathscr{F}}$ for every uniformly continuous function φ over $\mathbb{R}$.*

Proof. (a)$\Rightarrow$(b) Let φ be a uniformly continuous and monotonic function over $\mathbb{R}$. Let $I = \varphi(\mathbb{R})$ and suppose, for instance, that φ is nondecreasing. From the hypothesis, $\mathscr{F}$ satisfies the Lebesgue chain condition for f and for some $k > 0$. By Proposition 2.3, in order to prove that $\varphi \circ f \in \bar{\mathscr{F}}$ it is enough to show that $\mathscr{F}$ also satisfies the Lebesgue chain condition for $\varphi \circ f$ with the same k. Let $\{C_n\}_{n\in\mathbb{Z}}$ be a Lebesgue chain for $\varphi \circ f$ given by the sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$. For every $n \in \mathbb{Z}$ choose $\beta_n \in \bar{\mathbb{R}}$ such that $\varphi(\beta_n) = \alpha_n$ if $\alpha_n \in I$, $\beta_n = -\infty$ if $\alpha_n \notin I$ and $\alpha_n \leqslant \inf I$ and $\beta_n = \infty$ if $\alpha_n \notin I$ and $\alpha_n \geqslant \sup I$. Now, the sequence $\{\beta_n\}_{n\in\mathbb{Z}}$ is nondecreasing and it defines a Lebesgue chain $\{C'_n\}_{n\in\mathbb{Z}}$ for f. Indeed, if $r > 0$ is such that $\alpha_n - \alpha_{n-1} \geqslant r$ when α_n is a real number, then by the uniform continuity of φ there exists $s > 0$ with $\beta_n - \beta_{n-1} \geqslant s$ if β_n is a real number.

By (a), there exists $g \in \mathscr{F}$ with $|g(x) - n| < k$ if $x \in C'_n$. Since $C'_n \supset C_n$, it follows that $\mathscr{F}$ satisfies the Lebesgue chain condition for $\varphi \circ f$ with the same k and hence $\varphi \circ f \in \bar{\mathscr{F}}$.

(b)$\Rightarrow$(a) Let $k > 1$ and let $\{C_n\}_{n\in\mathbb{Z}}$ be a Lebesgue chain for f given by the sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ (thus, for some $r > 0$, $\alpha_n - \alpha_{n-1} \geqslant r$ when α_n is a real number). If for every $n \in \mathbb{Z}$, $\alpha_n = \pm\infty$, let p be such that $\alpha_p = -\infty$, $\alpha_{p+1} = \infty$ and take $h(x) = p + \frac{1}{2}$ for every $x \in X$. By hypothesis $h \in \bar{\mathscr{F}}$ and trivially $|h(x) - n| < 1$ if $x \in C_n$ $(n \in \mathbb{Z})$. Otherwise, if any member of the sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ is a real number, then for $t \in (\alpha_n, \alpha_{n+1}]$ we define (see Fig. 1)

$$\varphi(t) = \begin{cases} \dfrac{t - \alpha_n}{\alpha_{n+1} - \alpha_n} + n, & \text{if } \alpha_n \text{ and } \alpha_{n+1} \text{ are real numbers,} \\ n + 1, & \text{if } \alpha_n = -\infty, \\ n, & \text{if } \alpha_{n+1} = \infty. \end{cases}$$

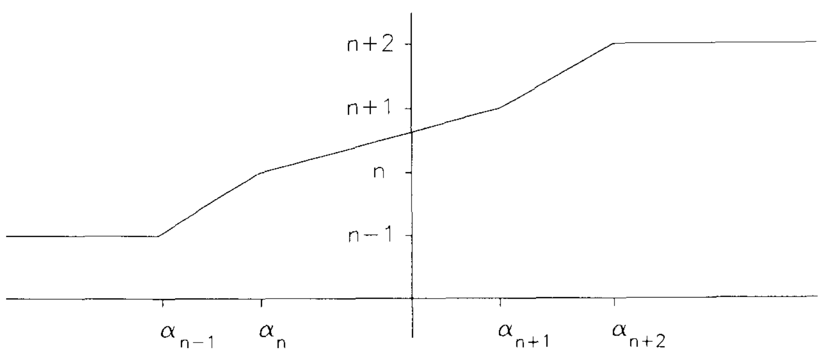

Fig. 1.

Thus φ is a nondecreasing function over $\mathbb{R}$ and $\varphi((\alpha_n, \alpha_{n+1}]) \subset [n, n+1]$ for every $n \in \mathbb{Z}$. On the other hand, it is easy to see that $|\varphi(t) - \varphi(t')| \leqslant (1/r)|t - t'|$ and therefore φ is a uniformly continuous function. From (b) the function $h = \varphi \circ f$ belongs to $\bar{\mathscr{F}}$, and the properties of φ imply that $|h(x) - n| \leqslant 1$ if $x \in C_n$ $(n \in \mathbb{Z})$.

Thus, as in the case that every α_n is a nonreal number, or in the other case, there is $h \in \bar{\mathscr{F}}$ with $|h(x) - n| \leqslant 1$ if $x \in C_n$ $(n \in \mathbb{Z})$. Finally, if $g \in \mathscr{F}$ satisfies $|h - g| < k - 1$, then $|g(x) - n| < k$ if $x \in C_n$ $(n \in \mathbb{Z})$.

(b)$\Leftrightarrow$(c) It is enough to prove (b)$\Rightarrow$(c). Since $\bar{\mathscr{F}}$ is a uniformly closed linear subspace, then from (b) it follows that $\varphi \circ f \in \bar{\mathscr{F}}$ for every φ in the uniformly closed linear subspace of $C(\mathbb{R})$ generated by the set of all uniformly continuous and monotonic functions. But this linear subspace is the set of all uniformly continuous functions over $\mathbb{R}$, as we shall see in the following lemma. □

Remark. In the above result we have implicitly shown that if $\mathscr{F}$ satisfies the Lebesgue chain condition for f with some constant k, then $\mathscr{F}$ also satisfies this condition for every constant $k > 1$.

4.2. Lemma. *Let $\mathscr{U}$ be the set of all uniformly continuous functions over $\mathbb{R}$ and let $\mathscr{F}$ be the linear subspace of $C(\mathbb{R})$ generated by the functions in $\mathscr{U}$ which are monotonic. Then $\bar{\mathscr{F}} = \mathscr{U}$.*

Proof. It follows at once that $\mathscr{U}$ is uniformly closed (if $f \in \bar{\mathscr{U}}$ and $\varepsilon > 0$ let $g \in \mathscr{U}$ with $|f - g| < \varepsilon/3$ and $\delta > 0$ such that $|g(t) - g(t')| < \varepsilon/3$ when $|t - t'| < \delta$ then $|f(t) - f(t')| < \varepsilon$ if $|t - t'| < \delta$) and hence $\bar{\mathscr{F}} \subset \mathscr{U}$. Conversely, if $f \in \mathscr{U}$ and $\varepsilon > 0$, let $\delta > 0$ be such that $|t - t'| \leqslant \delta$ implies $|f(t) - f(t')| \leqslant \varepsilon/2$. Consider the function $g: \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = (t - p\delta)\frac{f((p+1)\delta) - f(p\delta)}{\delta} + f(p\delta) \quad \text{if } t \in (p\delta, (p+1)\delta]\ (p \in \mathbb{Z}).$$

Thus, g is the polygonal which connects the points of $\mathbb{R}^2$ $(p\delta, f(p\delta))$, $p \in \mathbb{Z}$, and therefore its restriction to the interval $[p\delta, (p+1)\delta]$ is a straight line with slope $m_p = (f((p+1)\delta) - f(p\delta))/\delta$ and $|m_p| \leqslant \varepsilon/(2\delta)$. Now, it is obvious that for every $t, t' \in \mathbb{R}$, $|g(t) - g(t')| \leqslant (\varepsilon/(2\delta))|t - t'|$ and hence g is uniformly continuous. Moreover g belongs to $\mathscr{F}$ because the function $g_1(t) = g(t) + (\varepsilon/(2\delta))t$ is nondecreasing. Finally, it is easy to prove that $|g - f| < \varepsilon$ and this completes the proof. □

4.3. Definition. A subset $\mathscr{F}$ of $F(X)$ has the *property C*, whenever $\mathscr{F}$ satisfies the Lebesgue chain condition for every $f \in \mathscr{F}$.

From Proposition 4.1 or also from Theorem 2.7, it follows that every linear subspace which is uniformly dense in $C(X)$ has the property C. In the next theorem we shall see that the Lebesgue chain condition characterizes the uniform closure of linear subspaces with the property C.

4.4. Theorem. *Let $\mathscr{F} \subset F(X)$ a linear subspace over $\mathbb{R}$ with the property C and let $f \in F(X)$. Then $f \in \bar{\mathscr{F}}$ if and only if $\mathscr{F}$ satisfies the Lebesgue chain condition for f.*

Proof. The sufficiency follows from Proposition 2.3. Conversely, let $f \in \bar{\mathscr{F}}$. According to Proposition 4.1, if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{F}$ which converges to f and φ is a uniformly continuous function over $\mathbb{R}$, then $\varphi \circ f_n \in \bar{\mathscr{F}}$ for every $n \in \mathbb{N}$. From the uniform continuity of φ, it follows that the sequence $\{\varphi \circ f_n\}_{n\in\mathbb{N}}$ must converge uniformly to $\varphi \circ f$ and hence $\varphi \circ f \in \bar{\mathscr{F}}$. Again, from Proposition 4.1, $\mathscr{F}$ satisfies the Lebesgue chain condition for f. □

4.5. Theorem. *Let $\mathscr{F} \subset F(X)$ a linear subspace over $\mathbb{R}$. The following conditions are equivalent:*

(a) *$\mathscr{F}$ has the property C.*

(b) *$\bar{\mathscr{F}}$ is closed under composition with uniformly continuous and monotonic functions over $\mathbb{R}$.*

(c) *$\bar{\mathscr{F}}$ is closed under composition with uniformly continuous functions over $\mathbb{R}$.*

(d) *$\bar{\mathscr{F}}$ has the property C.*

(e) *There exists $k > 0$ such that $\mathscr{F}$ satisfies the Lebesgue chain condition for every $f \in \mathscr{F}$ and with the same k.*

Proof. It follows at once from Theorem 4.4, Proposition 4.1 and the Remark after Proposition 4.1. □

As in [2], we say that $\mathscr{F} \subset F(X)$ has the property S whenever $\mathscr{F}$ S-separates the Lebesgue sets of every f in $\mathscr{F}$.

4.6. Corollary (Blasco and Moltó [2], Mrowka [7]). *Let $\mathscr{F} \subset F^*(X)$ be a linear subspace over $\mathbb{R}$. The following conditions are equivalent:*

(a) *$\mathscr{F}$ has the property C.*

(b) *$\mathscr{F}$ has the property S.*

(c) *$\bar{\mathscr{F}}$ is a linear sublattice containing all the real constant functions.*

(d) *$\bar{\mathscr{F}}$ is a subring containing all the real constant functions.*

(e) *$\bar{\mathscr{F}}$ is closed under composition with continuous functions over $\mathbb{R}$.*

Proof. From Proposition 2.4, (a) is equivalent to (b). The equivalence among (c), (d) and (e) has been proved by Mrowka [7]. From Theorem 4.5, (e) implies (a). Finally (a) implies (c), because if $\mathscr{F}$ has the property C, then $\bar{\mathscr{F}}$ is closed under composition with all the real constant functions and with the function $\varphi(t) = |t|$ $(t \in \mathbb{R})$ (Theorem 4.5). □

For the unbounded case, most of the results in the above corollary are not true. Next, we shall compare the property C with the other conditions. (See [3] for more details).

It is very easy to prove that (a) always implies (b), but the converse is not true. From Theorem 4.5 it follows that (a) implies (c), but the converse is again not true. Moreover (a) does not imply either (d) or (e), (d) does not imply (a) and obviously (e) implies (a). The following examples prove all these assertions.

4.7. Examples. (1) The linear subspace $\mathscr{F}$ of $C(\mathbb{R})$ generated by $C^*(\mathbb{R})$ and the identity function is uniformly closed, it has the property S because it contains $C^*(\mathbb{R})$ but it has not the property C. (For instance, the function $g(x)=|x|$, $x\in\mathbb{R}$, is not in $\mathscr{F}$.) So $\mathscr{F}$ satisfies (b) but not (a).

(2) The set of all the uniformly continuous functions over $\mathbb{R}$ is a uniformly closed linear subspace with the property C, but it is not a subring. (For instance, the function $g(x)=x^2$, $x\in\mathbb{R}$, is not in $\mathscr{F}$.)

(3) The set of all the polynomials over $\mathbb{R}$ is a uniformly closed subring and it has not the property C. Since this set contains the identity function, if it has the property C then it must contain all the uniformly continuous functions (Theorem 4.5), but obviously that is not true.

(4) The set $\mathscr{F}$ of all functions $f\in C(\mathbb{R})$ for which there exists a straight line $ax+b$ with $\lim_{x\to\infty}(f(x)-(ax+b))=0$, is a uniformly closed linear sublattice containing all the real contant functions and it has not the property C. (For instance, the function $g(x)=x^{1/2}$ for $x\geqslant 0$, $g(x)=0$ for $x\leqslant 0$, is uniformly continuous and g does not belong to $\mathscr{F}$.)

Examples 4.7(3) and (4) show that for $\mathscr{F}$ being a uniformly closed linear subspace containing all the real constant functions neither being a ring nor a lattice is enough to conclude that $\mathscr{F}$ has the property C. Now, we shall prove that both of these conditions imply that $\mathscr{F}$ has the property C. For this we need the next lemma.

4.8. Lemma. *If φ is a uniformly continuous function over $\mathbb{R}$ and $t_0\in\mathbb{R}$, then there exist two straight lines a_1t+b_1 and a_2t+b_2 such that $\varphi(t)\leqslant a_1t+b_1$ if $t\geqslant t_0$ and $\varphi(t)\geqslant a_2t+b_2$ if $t\leqslant t_0$.*

Proof. Clearly, we can suppose $t_0=0$. From the uniform continuity of φ and for $\varepsilon=1$ there exists $\delta>0$ such that $|t-t'|<\delta$ implies $|\varphi(t)-\varphi(t')|\leqslant 1$.

If $t\geqslant 0$, there is a $p\in\mathbb{N}$ with $p\delta\leqslant t\leqslant(p+1)\delta$ and therefore $\varphi(t)\leqslant\varphi(p\delta)+1\leqslant\varphi((p-1)\delta)+2\leqslant\cdots\leqslant\varphi(0)+(p+1)$. Similarly, if $t\leqslant 0$, there is $q\in\mathbb{N}$ with $-(q+1)\delta\leqslant t\leqslant -q\delta$ and $\varphi(t)\geqslant\varphi(0)-(q+1)$.

Thus, the straight lines $(1/\delta)t+\varphi(0)+1$ and $(1/\delta)t+\varphi(0)-1$ satisfy the lemma. $\square$

4.9. Proposition. *If the uniform closure of a linear subspace $\mathscr{F}$ of $F(X)$ is a subring and a sublattice containing all the real constant functions, then $\mathscr{F}$ has the property C.*

Proof. Since $\mathscr{F}$ has the property C if and only if $\bar{\bar{\mathscr{F}}}$ has the property C (Theorem 4.5), then we can suppose that $\mathscr{F}$ is also uniformly closed. Denote $\mathscr{L}$ the set of the functions $\varphi \in C(\mathbb{R})$ such that $\varphi \circ f \in \mathscr{F}$ for every $f \in \mathscr{F}$. According to Theorem 4.5, $\mathscr{F}$ has the property C if and only if $\mathscr{L}$ contains the set of all uniformly continuous functions. Firstly, note that $\mathscr{L}$ inherits the algebraic and topological properties from $\mathscr{F}$. Moreover,

(1) *$\mathscr{L}$ contains the polynomials.* This is clear, because $\mathscr{L}$ is a subring containing all the real constant functions and the identity function.

(2) *$\mathscr{L}$ contains the polygonals with a finite number of vertices.* Indeed, since $\mathscr{L}$ is a sublattice containing the polynomials, then the polygonals with one vertex are in $\mathscr{L}$ because they are the supremum or the infimum of two lines. An easy induction (on the number of vertices) shows that $\mathscr{L}$ also contains all the polygonals with a finite number of vertices.

(3) *$\mathscr{L}$ contains the continuous functions with compact support.* This is an easy consequence of the above paragraph and of the classic Kakutani–Stone theorem. (This theorem provides conditions of uniform density for sublattices of continuous functions over a compact space.)

(4) *$\mathscr{L}$ contains $C_0(\mathbb{R})$* (i.e., the set of all continuous functions vanishing at infinity). It is enough to note that $C_0(\mathbb{R})$ is the uniform closure in $C(\mathbb{R})$ of the set of all continuous functions with compact support.

(5) *$\mathscr{L}$ contains the uniformly continuous functions.* By Lemma 4.2 it is sufficient to prove that if φ is uniformly continuous and monotonic (for instance nondecreasing), then $\varphi \in \mathscr{L}$. As in the above lemma, let $t_0 \in \mathbb{R}$, and let $a_1 t + b_1$ and $a_2 t + b_2$ be such that $\varphi(t_0) \leqslant \varphi(t) \leqslant a_1 t + b_1$ if $t \geqslant t_0$ and $\varphi(t_0) \geqslant \varphi(t) \geqslant a_2 t + b_2$ if $t \leqslant t_0$. Then

$$\frac{\varphi(t_0)}{1+t^2} \leqslant \frac{\varphi(t)}{1+t^2} \leqslant \frac{a_1 t + b_1}{1+t^2} \quad \text{if } t \geqslant t_0,$$

$$\frac{\varphi(t_0)}{1+t^2} \geqslant \frac{\varphi(t)}{1+t^2} \geqslant \frac{a_2 t + b_2}{1+t^2} \quad \text{if } t \leqslant t_0.$$

This implies that the function $\psi(t) = \varphi(t)/(1+t^2)$ belongs to $C_0(\mathbb{R})$, and hence $\varphi \in \mathscr{L}$, because it is the product of a function in $C_0(\mathbb{R})$ and a polynomial. $\square$

Throughout the above proof we have also shown that every uniformly closed subset of $F(X)$ which is a subring and sublattice containing all the real constant functions is also closed under composition with a large class of continuous functions over $\mathbb{R}$. For instance, it is closed under composition with polynomials, with functions in $C_0(\mathbb{R})$, with uniformly continuous functions, ..., and also with all bounded continuous functions, because if $\varphi \in C^*(\mathbb{R})$, then φ can be written like the product of the function $\varphi(t)/(1+t^2) \in C_0(\mathbb{R})$ and the polynomial $1+t^2$. But this large class of functions is not necessarily $C(\mathbb{R})$, as we shall prove in the next example.

4.10. Example. Let $\mathscr{F}$ be the subset of $C(\mathbb{R})$ defined by

$$\mathscr{F} = \left\{ \sum_{i=1}^{n} f_i p_i : f_i \in C_0(\mathbb{R}) \text{ and } p_i \text{ a polynomial, } i = 1, \ldots n,\ n \in \mathbb{N} \right\}.$$

(1) *$\mathcal{F}$ is the subring of $C(\mathbb{R})$ generated by $C_0(\mathbb{R})$ and the set of all the polynomials.* It is easy to check that $\mathcal{F}$ is a subring containing $C_0(\mathbb{R})$. Moreover, if $p(x)$ is a polynomial we can write

$$p(x) = \frac{p(x)}{(1+x^2)(p^2(x)+1)}(1+x^2)(p^2(x)+1)$$

and therefore $p(x)$ is the product of a function in $C_0(\mathbb{R})$ and a polynomial.

(2) *$\mathcal{F}$ is a sublattice.* If $f \in \mathcal{F}$, then

$$|f|(x) = \frac{|f(x)|}{(1+x^2)(f^2(x)+1)}(1+x^2)(f^2(x)+1).$$

Thus $|f| \in \mathcal{F}$, because it is also the product of a function in $C_0(\mathbb{R})$ and another function in $\mathcal{F}$.

(3) *$\mathcal{F}$ contains $C^*(\mathbb{R})$.* Again, by the same argument, if $f \in C^*(\mathbb{R})$, then $f \in \mathcal{F}$, because $f(x) = (f(x)/(1+x^2))(1+x^2)$.

(4) *$\mathcal{F}$ is uniformly closed.* Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ which converges to f. There exist $\nu \in \mathbb{N}$ such that $f - f_\nu$ is a bounded function and so $f \in \mathcal{F}$ because $f = (f - f_\nu) + f_\nu$.

We have proved that $\mathcal{F}$ is a uniformly closed subring and sublattice containing all the real constant functions, but obviously $\mathcal{F}$ is not $C(\mathbb{R})$ (for instance, $h(x) = \mathrm{e}^x$ is not in $\mathcal{F}$). Since $\mathcal{F}$ contains the identity function then $\mathcal{F}$ cannot be closed under composition with all continuous functions over $\mathbb{R}$.

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